CONNECTED ALGEBRAIC MONOIDS[1]

BY

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ABSTRACT. Let S be a connected algebraic monoid with group of units G and lattice of regular $\mathcal{J}$-classes $\mathcal{U}(S)$. The connection between the solvability of G and the semilattice decomposition of S into archimedean semigroups is further elaborated. If S has a zero and if $|\mathcal{U}(S)| \le 7$, then it is shown that G is solvable if and only if $\mathcal{U}(S)$ is relatively complemented. If $J \in \mathcal{U}(S)$, then we introduce two basic numbers $\theta(J)$ and $\delta(J)$ and study their properties. Crucial to this process is the theorem that for any idempotent e of S, the centralizer of e in G is connected. Connected monoids with central idempotents are also studied. A conjecture about their structure is forwarded. It is pointed out that the maximal connected submonoids of S with central idempotents need not be conjugate. However special maximal connected submonoids with central idempotents are conjugate. If S is regular, then S is a Clifford semigroup if and only if for all $f \in E(S)$, the set $\{e \mid e \in E(S), e \ge f\}$ is finite. Finally the maximal semilattice image of any connected monoid is determined.

Introduction. Let K be an algebraically closed field. By a (linear) algebraic semigroup, we mean a Zariski closed subsemigroup of the multiplicative semigroup of all $n \times n$ matrices over a field. An algebraic semigroup is connected if the underlying closed set is an irreducible variety. Connected groups have been well studied [1]. In this paper we continue our study of connected monoids. Our basic model of a connected monoid is the multiplicative semigroup of all $n \times n$ matrices, $\mathfrak{M}_n(K)$. Even if one is only interested in invertible matrices, it still becomes necessary to consider noninvertible matrices. In the same way, if a connected group is the group of units of a connected algebraic monoid, then it is only natural to study the monoid and see if its study sheds some light on the structure of the group. A basic concept in $\mathfrak{M}_n(K)$ is that of rank ρ. The correct generalization of rank to an abstract monoid S is given by the Green's relation $\mathcal{J}$ which is defined as: $a \mathcal{J} b$ if and only if $xay = b$, $sbt = a$ for some $x, y, s, t \in S$. The rank ordering in $\mathfrak{M}_n(K)$ generalizes to a partial ordering of the $\mathcal{J}$-classes of S. A $\mathcal{J}$-class with an idempotent is called regular. Let $\mathcal{U}(S)$ denote the set of all regular $\mathcal{J}$-classes of S. It turns out that $(\mathcal{U}(S), \le)$ is a lattice if S is a connected monoid. Let us now look at two basic connected monoids. Let $S_1 = \mathfrak{M}_n(K)$, $S_2 =$ the monoid of all upper triangular $n \times n$ matrices over K. Let G_i denote the group of units of S_i ($i = 1, 2$). G_2 is

solvable while G_1 is as far from being solvable as possible. Now let us compare $\mathfrak{U}(S_1)$ and $\mathfrak{U}(S_2)$. $\mathfrak{U}(S_1)$ is a linear chain while $\mathfrak{U}(S_2)$ is a Boolean lattice. The basic conjecture in this direction is the following.

CONJECTURE. Let S be a connected monoid with zero and group of units G. Then G is solvable if and only if $\mathfrak{U}(S)$ is a relatively complemented lattice. In this paper, we prove the conjecture when $|\mathfrak{U}(S)| \le 7$.

Let S be a connected monoid with groups of units G. The connection between the solvability of G and the semilattice decomposition of S into archimedean semigroups was pointed out in [9]. This point is further elaborated in the present paper. If $J \in \mathfrak{U}(S)$, then we introduce two basic numbers $\theta(J)$ and $\delta(J)$ and study their properties. Crucial to this process is the theorem that for any idempotent e of S, the centralizer of e in G is connected. If S is a connected regular monoid, then the set $\{e \mid e \in E(S),\ e \ge f\}$ is finite for all $f \in E(S)$, if and only if S is a Clifford semigroup. Connected monoids with central idempotents are also studied. A conjecture about their structure is forwarded. Let S be a connected monoid. Then it is pointed out that the maximal connected submonoids of S with central idempotents need not be conjugate. However special maximal connected submonoids with central idempotents are conjugate.

Let S be a connected monoid with group of units G and let T be a maximal torus of G. The maximal semilattice image of S is explicitly determined and is shown to be isomorphic to a subsemilattice of $(\mathfrak{U}(S), \wedge)$ and of $(E(\bar{T}), \cdot)$.

1. Preliminaries. Throughout this paper Z^+, $\mathbf{R}$ will denote the sets of positive integers and all reals, respectively. If X is a set then $|X|$ denotes the cardinality of X. If $(P, \le)$ is a partially ordered set, $\alpha, \beta \in P$, then β *covers* α if $\beta > \alpha$ and there is no $\gamma \in P$ such that $\beta > \gamma > \alpha$. A linearly ordered subset of P is a *chain*. The length of a chain $\{\alpha_1 < \cdots < \alpha_k\}$ is $k - 1$. A chain not contained in any other is a *maximal chain*. If $\xi \in P$, then ξ is a *minimal element* of P if there is no $\alpha \in P$ such that $\alpha < \xi$. ξ is a *minimum element* of P if $\xi \le \alpha$ for all $\alpha \in P$. Let $(L, \vee, \wedge)$ be a lattice. Then L is *relatively complemented* if for all $\alpha, \beta, \gamma \in L$, $\alpha > \beta > \gamma$ implies that there exists $\xi \in L$ such that $\beta \wedge \xi = \gamma$ and $\beta \vee \xi = \alpha$.

Let S be a monoid with group of units G. If $X, Y \subseteq S$, then X, Y are *conjugate* if $a^{-1}Xa = Y$ for some $a \in G$. If $X \subseteq S$ and $Y \subseteq G$, then $N_Y(X) = \{y \mid y \in Y, y^{-1}Xy = X\}$ is the *normalizer* of X in Y. If $X \subseteq S$, $Y \subseteq S$, then $C_Y(X) = \{y \mid y \in Y, xy = yx \text{ for all } x \in X\}$ is the *centralizer* of X in Y. $C(S) = C_S(S)$ is the *center* of S. Let S be a semigroup. If $a, b \in S$, then $a \mid b$ (a *divides* b) if $xay = b$ for some $x, y \in S^1$. S is *archimedean* if for all $a, b \in S$, $a \mid b^i$ for some $i \in Z^+$. Let S_α ($\alpha \in \Omega$) denote a partition of S into subsemigroups. Then S is a *semilattice* (*union*) of S_α ($\alpha \in \Omega$) if for all $\alpha, \beta \in \Omega$, there exists $\gamma \in \Omega$ such that $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$. By the author [4], S is a semilattice of archimedean semigroups if and only if for all $a, b \in S$, $a \mid b$ implies $a^2 \mid b^i$ for some $i \in Z^+$. A semigroup with no proper semilattice decomposition is said to be $\mathfrak{S}$-*indecomposable*. By Tamura [10, 11, 12] every semigroup is a semilattice of $\mathfrak{S}$-indecomposable semigroups. We let $E(S)$ denote the set of all idempotents of S. If $X \subseteq S$, then $E(X) = E(S) \cap X$. If $e, f \in E(S)$, then $e \ge f$ if $ef = fe = f$. We write $E(S)$ for the partially ordered set $(E(S), \le)$. $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$,

$\mathcal{K}$, $\mathcal{D}$ will denote the usual Green's relations on S. If $a \in S$, then H_a, J_a will denote the $\mathcal{K}$-class and $\mathcal{J}$-class of a, respectively. We let $\mathcal{U}(S) = \{ J \mid J$ is a $\mathcal{J}$-class of S and $E(J) \neq \varnothing \}$. If $J_1, J_2 \in \mathcal{U}(S)$, then $J_1 \geqslant J_2$ if $a_1 \mid a_2$ for some (all) $a_i \in J_i$, $i = 1, 2$. We write $\mathcal{U}(S)$ for the partially ordered set $(\mathcal{U}(S), \leqslant)$.

K will denote a fixed algebraically closed field. If $n \in Z^+$, then $K^n = K \times \cdots \times K$ is the affine n-space and $\mathcal{M}_n(K)$ the multiplicative monoid of all $n \times n$ matrices over K. If $X \subseteq K^n$, then X is *closed* if X is the set of zeros of some polynomials on K^n. If $X \subseteq K^n$, the $\overline{X}$ denotes the *closure* of X. By an *algebraic semigroup* we mean a closed subset of K^n along with an associative operation which is a polynomial map. By a *homomorphism* between algebraic semigroups we mean a semigroup homomorphism which is also a morphism of varieties ($=$ a polynomial map). By a *connected semigroup* we mean an algebraic semigroup such that the underlying closed set is irreducible (i.e., is not a union of two proper closed subsets). Let S be an algebraic monoid with group of units G. Then S is isomorphic to a closed submonoid of $\mathcal{M}_n(K)$ for some $n \in Z^+$. A power of each element of S lies in a subgroup of S, $\mathcal{J} = \mathcal{D}$, and $\mathcal{U}(S)$ is finite. If $J \in \mathcal{U}(S)$, then J^0 is completely 0-simple. G is a principal open subset of S and can therefore be viewed as an algebraic group. $S \setminus G$ is a closed prime ideal of S. If S_1 is a closed submonoid of S, then $G_1 = S_1 \cap G$ is the group of units of S_1. The identity element 1 of S lies in a unique irreducible component S^c of S. Moreover S^c is a closed connected submonoid of S and $S^c \cap G = G^c$. If S_1 is a closed submonoid of S then $S_1^c \cap G = (S_1 \cap G)^c$. If S is connected, then so is G, and in that case $\overline{G} = S$. If $e \in E(S)$, then H_e is the group of units of eSe. So H_e is an algebraic group. If S is connected, then so is H_e and $\overline{H_e} = eSe$.

We will need a number of basic results on (linear) algebraic groups. See [1, 3] for the extensive literature on algebraic groups. Let G be a connected algebraic group and let T be a maximal torus of G. We let $W(G) = N_G(T)/C_G(T)$ denote the *Weyl group* of G. Then [3, Proposition 24.1A, Corollary 25.2C], $W(G)$ is a finite group, and G is solvable if and only if $\mid W(G) \mid = 1$.

LEMMA 1.1. *Let S be an algebraic monoid, S_1, S_2 closed submonoids of S. Then $(S_1^c \cap S_2^c)^c = (S_1^c \cap S_2)^c = (S_1 \cap S_2)^c$.*

PROOF. $S_1^c \cap S_2^c \subseteq S_1^c \cap S_2 \subseteq S_1 \cap S_2$. Hence $(S_1^c \cap S_2^c) \subseteq (S_1^c \cap S_2)^c \subseteq (S_1 \cap S_2)^c \subseteq S_1^c \cap S_2^c$. This yields the result.

LEMMA 1.2. *Let S be an algebraic monoid with group of units G. Then*
(1) *If $X, Y \subseteq S$, then $\overline{XY} = \overline{\overline{X}\overline{Y}}$.*
(2) *If $X \subseteq S$, $a, b \in G$, then $\overline{aXb} = a\overline{X}b$.*
(3) *If $X \subseteq S$, $Y \subseteq G$, then $N_Y(X) \subseteq N_Y(\overline{X})$.*
(4) *If $X, Y \subseteq S$, then $C_Y(X) = C_Y(\overline{X})$.*

PROOF. (1) The set $\{x \mid x \in S, xY \subseteq \overline{XY}\}$ is closed and contains X. Hence $\overline{X}Y \subseteq \overline{XY}$. Similarly $\overline{X}\overline{Y} \subseteq \overline{XY}$. Since $XY \subseteq \overline{X}\overline{Y}$, $\overline{XY} = \overline{\overline{X}\overline{Y}}$.

(2) Since $a, b \in G$, $a\overline{X}b$ is closed. Since $aXb \subseteq a\overline{X}b$, $\overline{aXb} \subseteq a\overline{X}b$. Similarly, since $X \subseteq a^{-1}\overline{aXb}b^{-1}$, we see that $\overline{X} \subseteq a^{-1}\overline{aXb}b^{-1}$.

(3) Let $a \in N_Y(X)$. Then $a^{-1}Xa = X$. By (2), $a^{-1}\overline{X}a = \overline{X}$.

(4) Let $y \in C_Y(X)$. Then $C_S(y)$ is a closed set containing X. Hence $\overline{X} \subseteq C_S(y)$. So $y \in C_Y(\overline{X})$.

LEMMA 1.3. *Let S be a connected monoid with group of units G. Let T be a maximal torus of G, $e \in E(\overline{T})$. Let $S_1 = \{a \mid a \in S, ae = ea = e\}$, $T_1 = \{a \mid a \in T, ae = ea = e\}$, $S_2 = S_1^c$, $T_2 = T_1^c$, G_2 the group of units of S_2. Then*

(1) *for all $f \in E(S), f \geqslant e$ implies $f \in S_2$;*

(2) *T_2 is a maximal torus of G_2;*

(3) *if e is a minimal idempotent of S, then $E(\overline{T}) = E(\overline{T}_2)$.*

PROOF. (1) By [7, Lemma 1.1], there exists a closed connected submonoid of S' of S such that f is the zero of S'. Then $S' \subseteq S_1$. Hence $S' \subseteq S_1^c = S_2$. So $f \in S_2$.

(2) Let $\Gamma = \{1 > e_1 > \cdots > e_k = e > e_{k+1} > \cdots > e_m\}$ be a maximal chain in $E(\overline{T})$. Then by [9, Lemma 17], Γ is a maximal chain in $E(S)$. Let $\Gamma_0 = \{1 > e_1 > \cdots > e_k = e\}$. Then by (1) (applied to $\overline{T}$), $\Gamma_0 \subseteq E(\overline{T}_2)$. Clearly Γ_0 is a maximal chain in $E(S_2)$. Since e is the zero of $\overline{T}_2$, we see by [6, Theorem 3.17] that $\dim T_2 = k$. Let $T_2 \subseteq T_3$ where T_3 is a maximal torus of G_2. Then clearly Γ_0 is a maximal chain in $E(\overline{T}_3)$ and e is the zero of $\overline{T}_3$. So again by [6, Theorem 3.17], $\dim T_3 = k$. So $T_2 = T_3$.

(3) If e is a minimal idempotent of S, then clearly e is the minimum idempotent of $\overline{T}$ and so by (1) (applied to $\overline{T}$), $E(\overline{T}) \subseteq E(\overline{T}_2)$.

LEMMA 1.4. *Let S be a connected monoid with group of units G. Let T be a maximal torus of G, $e \in E(\overline{T})$. Then eT is a maximal torus of H_e.*

PROOF. Clearly $T \subseteq C_G(e)^c$. Consider the homomorphism, $\phi: C_G(e)^c \to H_e$ given by $\phi(a) = ea$. By [8, Theorem 4], ϕ is surjective. Hence by [3, Corollary 21.3C], $\phi(T) = eT$ is a maximal torus of H_e.

LEMMA 1.5. *Let S be a connected monoid with group of units G. Let T be a maximal torus of G. Let $e, f \in E(S)$ such that $e \mid f$. Then*

(1) *if $e \in E(\overline{T})$, there exists $f' \in E(\overline{T})$ such that $e \geqslant f'$ and $f \mathscr{J} f'$;*

(2) *if $f \in E(\overline{T})$, there exists $e' \in E(\overline{T})$ such that $e' \geqslant f$ and $e' \mathscr{J} e$.*

PROOF. (1) By [6, Lemma 1.3], there exists $f_1 \in E(S)$ such that $f \mathscr{J} f_1$ and $e \geqslant f_1$. By Lemma 1.4, eT is a maximal torus of H_e. Since H_e is the group of units of eSe and since $f_1 \in eSe$, we see [7, Theorem 1.8] that there exists $f' \in E(\overline{eT})$ such that $f_1 \mathscr{J} f'$. So $f' \in E(\overline{T})$, $e \geqslant f'$.

(2) By [8, Theorem 10], there exists $e_1 \in E(S)$ such that $e \mathscr{J} e_1$ and $e_1 \geqslant f_1$. Let $S_1 = \{a \mid a \in S, af = fa = f\}$, $T_1 = \{a \mid a \in T, af = f\}$, $S_2 = S_1^c$, $T_2 = T_1^c$, G_2 be the group of units of S_2. Then by Lemma 1.3, $e_1, f \in S_2$ and T_2 is a maximal torus of G_2. So by [7, Theorem 1.8] there exists $e' \in E(\overline{T}_2)$ such that $e_1 \mathscr{J} e'$. Then $e' \geqslant f$, $e' \in E(\overline{T})$.

LEMMA 1.6. *Let S be a connected monoid with group of units G. Let T be a maximal torus of G and let $e, f \in E(\overline{T})$, $e > f$. Then the following conditions are equivalent:*

(1) *e covers f in $E(S)$.*

(2) *e covers f in $E(\overline{T})$.*

(3) *J_e covers J_f in $\mathscr{U}(S)$.*

PROOF. That (1) $\Rightarrow$ (2) is obvious. So assume (2). Suppose J_e does not cover J_f. Then there exists $e_1 \in E(S)$ such that $e_1 \notin J_e$, $e_1 \notin J_f$ and $e \mid e_1 \mid f$. By [6, Lemma 1.3], there exists $e_2 \in E(S)$ such that $e > e_2$, $e_2 \mathcal{J} e_1$. So $e_2, f \in eSe$. By Lemma 1.4, eT is a maximal torus of H_e. By [6, Lemma 1.7], $e_2 \mid f$ in eSe. So by Lemma 1.5, there exists $e_3 \in E(e\overline{T})$ such that $e_3 \mathcal{J} e_2$ and $e_3 > f$. Then $e > e_3 > f$, $e_3 \in E(\overline{T})$. This contradiction shows that (2) $\Rightarrow$ (3). Next assume (3). Suppose $e > e_1 > f$ for some $e_1 \in E(S)$. Then $J_e > J_{e_1} > J_f$, a contradiction. Hence (3) $\Rightarrow$ (1).

LEMMA 1.7. *Let S be a connected monoid with group of units G and let T be a maximal torus of G.*

(1) *If $J \in \mathcal{U}(S)$, $e \in J \cap E(\overline{T})$, then $J \cap E(\overline{T}) = \{x^{-1}ex \mid x \in N_G(T)\}$.*

(2) *If $J_1, J_2 \in \mathcal{U}(S)$, then $J_1 > J_2$ if and only if there exist $e_i \in J_i \cap E(\overline{T})$, $i = 1, 2$, such that $e_1 > e_2$.*

PROOF. (1) Let $f \in E(\overline{T}) \cap J$. Then by [8, Theorem 9], $x^{-1}ex = f$ for some $x \in G$. Since $T \subseteq C_G(f)$, $xTx^{-1} \subseteq C_G(e)$. So T, xTx^{-1} are maximal tori of $C_G(e)^c$. Hence there exists $y \in C_G(e)^c$ such that $y^{-1}Ty = xTx^{-1}$. Then $x^{-1}y^{-1}Tyx = T$. So $yx \in N_G(T)$. Since $y \in C_G(e)$, $(yx)^{-1}eyx = x^{-1}ex = f$. This proves (1).

(2) By [7, Theorem 1.8], there exists $e_1 \in J_1 \cap E(\overline{T})$. If $J_1 > J_2$, then by Lemma 1.5, there exists $e_2 \in J_2 \cap E(\overline{T})$ such that $e_1 > e_2$. The converse is trivial.

LEMMA 1.8. *let S be a connected monoid, $e \in E(S)$. Then $\mathcal{V} = \{J \mid J \in \mathcal{U}(S), J \leqslant J_e\}$ is a sublattice of $\mathcal{U}(S)$ and $\mathcal{U}(eSe) = \{J \cap eSe \mid J \in \mathcal{V}\} \cong \mathcal{V}$.*

PROOF. Let $J \in \mathcal{V}$, $f \in E(J)$. Then $e \mid f$. By [6, Lemma 1.3], $e \geqslant f'$ for some $f' \in E(J)$. So $f' \in J \cap eSe$. It follows by [6, Lemma 1.7] that $J \cap eSe \in \mathcal{U}(eSe)$. The rest follows from [6, Lemma 1.7].

COROLLARY 1.9. *Let S be a connected monoid, $e \in E(S)$. If $\mathcal{U}(S)$ is a relatively complemented lattice, then so is $\mathcal{U}(eSe)$.*

LEMMA 1.10. *Let S be a connected monoid, $e \in E(S)$. Suppose eSe is a semilattice of archimedean semigroups. Let $f_1, f_2 \in E(S)$ such that $e \geqslant f_1$, $e \geqslant f_2$, $f_1 \mathcal{J} f_2$ and $f_1 f_2 = f_2 f_1$. Then $f_1 = f_2$.*

PROOF. Let $f_1, f_2 \in J \in \mathcal{U}(S)$. Then by Lemma 1.8, $f_1, f_2 \in J \cap eSe \in \mathcal{U}(eSe)$. By [9, Proposition 21] $J \cap eSe$ is completely simple. Since $f_1 f_2 = f_2 f_1$ we see that $f_1 = f_2$.

LEMMA 1.11. *Let S be a closed connected subsemigroup of $\mathfrak{M}_n(K)$ and let $J \in \mathcal{U}(S)$. Suppose that for all $e, f \in E(J)$, any eigenvalue of ef is either 0 or 1. Then $J^2 \subseteq J$.*

PROOF. By the proof of [9, Proposition 23], $E(J)^2 \subseteq E(J)$. Since J^0 is completely 0-simple, it follows easily that $J^2 \subseteq J$.

LEMMA 1.12. *Let S be a connected semigroup, $J \in \mathcal{U}(S)$. If $x, y \in J$, then there exist $e_1, e_2 \in E(J)$ such that $x \mathcal{R} e_1 \mathcal{L} e_2 \mathcal{R} y$.*

PROOF. J^0 is completely 0-simple, and hence admits a Rees representation $J^0 = (\Gamma \times G \times \Lambda) \cup \{0\}$ with sandwich map $P: \Lambda \times \Gamma \to G^0$ where G is a group.

Multiplication in J^0 is given by

$$(\alpha, a, \beta)(\gamma, b, \delta) = \begin{cases} (\alpha, aP(\beta, \gamma)b, \delta) & \text{if } P(\beta, \gamma) \neq 0, \\ 0 & \text{if } P(\beta, \gamma) = 0. \end{cases}$$

Let $x = (\alpha_1, a_1, \beta_1)$, $y = (\alpha_2, a_2, \beta_2)$. By [5, Theorem 2.16] there exists $\mu \in \Lambda$ such that $P(\mu, \alpha_1) \neq 0$, $P(\mu, \alpha_2) \neq 0$. Let $e_1 = (\alpha_1, P(\mu, \alpha_1)^{-1}, \mu)$, $e_2 = (\alpha_2, P(\mu, \alpha_2)^{-1}, \mu)$. Then $e_1, e_2 \in E(J)$ and $x \mathcal{R} e_1 \mathcal{L} e_2 \mathcal{R} y$.

LEMMA 1.13. *Let S be a commutative connected monoid, $e \in E(S)$. If Ω is a finite group of automorphisms of S having e as a common fixed point, then there exists a closed connected submonoid S_1 of S such that $e \in S_1$ and $\phi(a) = a$ for all $\phi \in \Omega$, $a \in S_1$.*

PROOF. Let $\Omega = \{\phi_1, \ldots, \phi_k\}$ and define $\psi: S \to S$ as $\psi(x) = \phi_1(x) \cdots \phi_k(x)$. Since S is commutative, ψ is a homomorphism. Clearly $\psi(1) = 1$ and $\psi(e) = e$. Let $\phi \in \Omega$. Then since $\phi\Omega = \Omega$, $\phi\psi(x) = \psi(x)$ for all $x \in S$. Let $S_1 = \overline{\psi(S)}$. Then S_1 is a closed connected submonoid of S and $e \in S_1$. Clearly $\phi(a) = a$ for all $\phi \in \Omega$ and $a \in S_1$.

PROPOSITION 1.14. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Let $\mathcal{X} = \{J \mid J \in \mathfrak{U}(S), G \text{ covers } J\}$, $\mathcal{Y} = \{J \mid J \in \mathfrak{U}(S), J_1 \text{ covers } J \text{ for some } J_1 \in \mathcal{X}\}$, $X = \{e \mid e \in E(\overline{T}), 1 \text{ covers } e\}$, $Y = \{f \mid f \in E(\overline{T}), e \text{ covers } f \text{ for some } e \in X\}$. Then*
  (1) $X = \bigcup_{J \in \mathcal{X}} E(\overline{T}) \cap J$.
  (2) $Y = \bigcup_{J \in \mathcal{Y}} E(\overline{T}) \cap J$.
  (3) *If $J_1 \in \mathfrak{U}(S)$, $J_2 \in \mathcal{Y}$ and if $G > J_1 > J_2$, then $J_1 \in \mathcal{X}$ and J_1 covers J_2.*
  (4) *If $e \in E(\overline{T})$, $f \in Y$, and if $1 > e > f$, then e covers f and $e \in X$.*
  (5) *If $J' \in \mathcal{Y}$, then $|\{J \mid J \in \mathcal{X}, J > J'\}| \leqslant 2$.*
  (6) *If $f \in Y$, then $|\{e \mid e \in X, e > f\}| \leqslant 2$.*
  (7) *If $\mathfrak{U}(S)$ is relatively complemented, if $f \in Y$, $e_1, e_2 \in X$, $e_1 > f$, $e_2 > f$, and if $e_1 \mathcal{J} e_2$, then $e_1 = e_2$.*

PROOF. (1), (2) follow from Lemmas 1.5 and 1.6. Next let $J_2 \in \mathcal{Y}$, $J_1 \in \mathfrak{U}(S)$ such that $G > J_1 > J_2$. There exists $J \in \mathcal{X}$ such that J covers J_1. So we have a maximal chain $\Gamma = \{G > J > J_2 > J_3 > \cdots > J_t\}$ in $\mathfrak{U}(S)$. Then $\Lambda = \{G > J_1 > J_2 > J_3 > \cdots > J_t\}$ is a chain in $\mathfrak{U}(S)$. Since by [7, Theorem 1.9] all maximal chains in $\mathfrak{U}(S)$ have the same length, Λ is also a maximal chain in $\mathfrak{U}(S)$. So G covers J_1 and J_1 covers J_2. This proves (3). Now (4) follows from Lemma 1.6.

We now prove (5). Let J_1, J_2, J_3 be distinct elements of $\mathcal{X}$, $J' \in \mathcal{Y}$ such that $J_i > J'$, $i = 1, 2, 3$. Let $f \in J' \cap E(\overline{T})$. By Lemma 1.5, there exists $e_i \in J_i \cap E(\overline{T})$, $i = 1, 2, 3$, such that $e_i > f$, $i = 1, 2, 3$. Then by (1), (2), $e_1, e_2, e_3 \in X$ and $f \in Y$. Let $T_1 = \{a \mid a \in T, af = f\}$, $T_2 = T_1^c$. Then by Lemma 1.3, $e_1, e_2, e_3, f \in \overline{T}_2$. Clearly f is the zero of $\overline{T}_2$. By (4), $\{1 > e_1 > f\}$ is a maximal chain in $E(\overline{T}_2)$. By [6, Theorem 3.17], $\dim \overline{T}_2 = 2$. So by [6, Theorem 3.18], $|E(\overline{T}_2)| \leqslant 4$. However, $1, e_1, e_2, e_3, f$ are distinct elements of $E(\overline{T}_2)$. This contradiction proves (5). We now obtain (6) by applying (5) to $\overline{T}$.

Next we prove (7). Suppose $\mathfrak{A}(S)$ is relatively complemented, $e_1, e_2 \in X$, $e_1 \,\mathcal{J}\, e_2$, $e_1 \neq e_2$, $e_1 > f$, $e_2 > f$. Let J_1 denote the $\mathcal{J}$-class of e_1, J' the $\mathcal{J}$-class of f. Then $J_1 \wedge J = J'$, $J_1 \vee J = G$ for some $J \in \mathfrak{A}(S)$. By (1), (2), (3), $J_1, J \in \mathcal{X}$ and $J' \in \mathcal{Y}$. By Lemma 1.5, there exists $e_3 \in J \cap E(\overline{T})$ such that $e_3 > f$. By (1), $e_3 \in X$. So $e_i > f$, $i = 1, 2, 3$. This contradicts (6), proving (7).

LEMMA 1.15. *Let S be a connected monoid such that the group of units T of S is a torus. Let $X = \{e \mid e \in E(S), 1 \text{ covers } e\}$, $Y = \{f \mid f \in E(S), e \text{ covers } f \text{ for some } e \in X\}$. Then for any $e, e' \in X$ with $e \neq e'$, there exist distinct $f_1, \ldots, f_k \in Y$ and distinct $e_1, \ldots, e_{k+1} \in X$ such that $e = e_1$, $e' = e_{k+1}$ and $e_i > f_i$, $e_{i+1} > f_i$, $i = 1, \ldots, k$.*

PROOF. By Lemma 1.3, we can assume that S has a zero. If P is a polytope in $\mathbf{R}^n$, then let $\mathcal{F}(P)$ denote the lattice of all faces of P (see [2]). By [7, Theorem 3.6], $E(S)$ is isomorphic to $\mathcal{F}(P)$ for a polytope P in some $\mathbf{R}^n$. Let P^x denote the dual polytope of P [2, §3.4, p. 46]. Then $\mathcal{F}(P)$ is anti-isomorphic to $\mathcal{F}(P^*)$. Let $\phi\colon E(S) \to \mathcal{F}(P^*)$ denote the corresponding map. Then $\phi(X)$ consists of all vertices of P^* and $\phi(Y)$ consists of all edges of P. By a result of Balinski (see [2, §11.3, Theorem 2]) there exist vertices $u_1, \ldots, u_{k+1}$ and edges $v_1, \ldots, v_k$ such that $\phi(e) = u_1$, $\phi(e') = u_{k+1}$ and u_i, u_{i+1} are the vertices of v_i, $i = 1, \ldots, k$. Let $e_i = \phi^{-1}(u_i)$, $i = 1, \ldots, k+1$, $f_i = \phi^{-1}(v_i)$, $i = 1, \ldots, k$. Then $e_i > f_i$, $e_{i+1} > f_i$, $i = 1, \ldots, k$. If we choose k to be minimal, then clearly, $e_1, \ldots, e_{k+1}$ and $f_1, \ldots, f_k$ will be distinct.

LEMMA 1.16. *Let G be a connected group, H a closed connected normal subgroup of G, B_0 a Borel subgroup of G such that $B_0 H = G$. Then $|W(G)| = |W(H)|$.*

PROOF. Since $H \lhd G$ and since all Borel subgroups of G are conjugate, $BH = G$ for all Borel subgroups B of G. Let U be a Borel subgroup of H. Then $U \subseteq V$ for some Borel subgroup V of G. Then $U \subseteq V \cap H$ and $V \cap H$ is solvable. By [3, Corollary 23.1A] $V \cap H = U$ is a Borel subgroup of H. Since $H \lhd G$, $B \cap H$ is a Borel subgroup of H for any Borel subgroup B of G. Let T_1 be a maximal torus of H and let $\mathcal{Q} = \{B \mid B \text{ is a Borel subgroup of } G, T_1 \subseteq B\}$, $\mathcal{Q}_1 = \{B \cap H \mid B \in \mathcal{Q}\}$. Then by [3, Proposition 24.1A], $|\mathcal{Q}_1| = |W(H)|$. Let $B_1, B_2 \in \mathcal{Q}$ such that $B_1 \cap H = B_2 \cap H$. Now $x^{-1}B_1 x = B_2$ for some $x \in G = B_1 H$. So $x = bh$ for some $b \in B_1$, $h \in H$. So $h^{-1}B_1 h = B_2$. Thus $h^{-1}[B_1 \cap H]h = B_2 \cap H = B_1 \cap H$. So $h \in N_H(B_1 \cap H)$. By [3, Theorem 23.1], $N_H(B_1 \cap H) = B_1 \cap H$. Thus $h \in B_1 \cap H \subseteq B_1$, and $B_1 = B_2$. It follows that $|\mathcal{Q}| = |\mathcal{Q}_1| < \infty$. Hence T_1 is a regular torus of G (see [3, Chapter IX]). Hence [3, Proposition 24.2], $|\mathcal{Q}| = |W(G)|$.

COROLLARY 1.17. *Let G be a connected group, H a closed normal subgroup of G, B a Borel subgroup of G such that $BH = G$. Then $|W(G)| = |W(H^c)|$.*

PROOF. Since $H \lhd G$, $H^c \lhd G$. So BH^c is a subgroup of G. Since BH^c is the image of the product map from $B \times H^c$ into G, BH^c is a constructible subgroup of G. Hence [3, Proposition 7.4A], BH^c is a closed subgroup of G. By [3, Proposition 7.3], $H = H^c h_1 \cup \cdots \cup H^c h_k$ for some $h_1, \ldots, h_k \in H$. Hence $G = BH = BH^c h_1 \cup \cdots \cup BH^c h_k$. Since G is connected, $G = BH^c h_i$ for some i. So $h_i \in BH^c$ and $G = BH^c$. We are now done by Lemma 1.16.

COROLLARY 1.18. *Let* $\phi: G \to G'$ *be a surjective homomorphism of connected groups such that* G' *is solvable and let* H *denote the kernel of* ϕ. *Then* $|W(G)| = |W(H^c)|$.

PROOF. Let B be a Borel subgroup of G. Then by [3, Corollary 21.3C], $\phi(B) = G'$. So $G = BH$. We are done by Corollary 1.17.

LEMMA 1.19. *Let* $\phi: G \to G'$ *be a surjective homomorphism of connected groups and let* H *denote the kernel of* ϕ. *Then* $|W(G)| = |W(H^c)| |W(G')|$.

PROOF. Let T be a maximal torus of G. Then by [3, Corollary 21.3C], $T' = \phi(T)$ is a maximal torus of G'. Let $\mathcal{C} = \{B \mid B$ is a Borel subgroup of G, $T \subseteq B\}$, $\mathcal{C}' = \{B' \mid B'$ is a Borel subgroup of G', $T' \subseteq B'\}$. By [3, Chapter IX], $|\mathcal{C}| = |W(G)|$, $|\mathcal{C}'| = |W(G')|$. If $B \in \mathcal{C}$, then by [3, Corollary 21.3C], $\phi(B) \in \mathcal{C}'$. If $B' \in \mathcal{C}'$, then let $\mathcal{C}_{B'} = \{B \mid B \in \mathcal{C}, \phi(B) = B'\}$. Then clearly $|\mathcal{C}| = \Sigma_{B' \in \mathcal{C}'} |\mathcal{C}_{B'}|$. Let $B' \in \mathcal{C}'$. It suffices to show that $|W(H^c)| = |\mathcal{C}_{B'}|$. Let $G_1 = \phi^{-1}(B')^c$. Then $T \subseteq G_1$. By [3, Corollary 21.3C], $\phi: G_1 \to B'$ is surjective. Let H_1 denote the kernel of this map. Then since $H^c \subseteq G_1$, $H^c = H_1^c$. Let $\mathcal{C} = \{B \mid B$ is a Borel subgroup of G_1, $T \subseteq B\}$. Let $B \in \mathcal{C}$. By [3, Corollary 21.3C], $\phi(B) = B'$. Let $B \subseteq B_1$, B_1 a Borel subgroup of G. Then $B' \subseteq \phi(B_1)$ and $\phi(B_1)$ is a Borel subgroup of G'. Thus $B' = \phi(B_1)$. Hence $B_1 \subseteq G_1$ and $B = B_1 \in \mathcal{C}_{B'}$. So $\mathcal{C} = \mathcal{C}_{B'}$. Thus $|\mathcal{C}_{B'}| = |W(G_1)|$. By Lemma 1.18, $|W(G_1)| = |W(H^c)|$. This proves the lemma.

COROLLARY 1.20. *Let* S *be a connected monoid with group of units* G, $e \in E(C(S))$. *Let* $G_1 = \{a \mid a \in G, ae = e\}$. *Then* $|W(G)| = |W(G_1^c)| |W(H_e)|$.

PROOF. The homomorphism $\phi: G \to H_e$ is given by $\phi(x) = ex$ is surjective by [8, Lemma 3]. Clearly G_1 is the kernel of ϕ. We are done by Lemma 1.19.

LEMMA 1.21. *Let* S *be an* $\mathcal{S}$-*indecomposable semigroup*, $J \in \mathcal{U}(S)$. *Then* $J^2 \subseteq J$ *if and only if* J *is the kernel of* S.

PROOF. By Tamura [10], $S^1 J S^1$, being an ideal of S, is $\mathcal{S}$-indecomposable. If J is not the kernel of S, then $\{J, S^1 J S^1 \setminus J\}$ is a proper semilattice decomposition of $S^1 J S^1$, a contradiction.

2. Main section. Fundamental in the theory of connected algebraic monoids is the following theorem (see [7, Theorems 1.8, 3.6, Corollary 1.6]; [6, Theorem 3.20]).

THEOREM 2.1 [7]. *Let* S *be a connected monoid with group of units* G *and let* T *be a maximal torus of* G. *Then*

(1) $\mathcal{U}(S)$ *is a finite lattice.*

(2) $E(\overline{T})$ *is a finite, relatively complemented lattice (in fact the lattice of all faces of a rational polytope in some* $\mathbf{R}^n$).

(3) *Any maximal chain in* $\mathcal{U}(S)$ *has the same length as any maximal chain in* $E(\overline{T})$ *or* $E(S)$.

(4) $E(S) = \bigcup_{x \in G} x^{-1} E(\overline{T}) x$.

See [8, Theorem 3] for the following theorem.

THEOREM 2.2 [8]. *Let S be a connected monoid with group of units G. If $J \in \mathfrak{U}(S)$, and if $a, b \in J$, then $xay = b$ for some $x, y \in G$. If $e, f \in E(J)$, then $x^{-1}ex = f$ for some $x \in G$.*

A special case of the following basic result was proved in [9, Theorem 14].

THEOREM 2.3. *Let S be a connected monoid with group of units G. Then for any $e \in E(S)$, $C_G(e)$ is a connected group.*

PROOF. By Theorem 2.1, $e \in \overline{T}$ for some maximal torus T of G. First let $u \in C_G(e) \cap N_G(T)$. Since $N_G(T)/C_G(T)$ is a finite group [3, Chapter IX], $u^i \in C_G(T)$ for some $i \in Z^+$. Consider the automorphism $\phi: \overline{T} \to \overline{T}$ given by $\phi(x) = u^{-1}xu$. Since $u^i \in C_G(T) = C_G(\overline{T})$, ϕ^i is the identity map. Since $u \in C_G(e)$, $\phi(e) = e$. By Lemma 1.13 there exists a subtorus T_1 of T such that $e \in \overline{T}_1$ and $\phi(a) = a$ for $a \in T_1$. So $u \in C_G(T_1)$. By [3, Theorem 22.3], $C_G(T_1)$ is connected. On the other hand, since $e \in \overline{T}_1$, $C_G(T_1) = C_G(\overline{T}_1) \subseteq C_G(e)$. So $C_G(T_1) \subseteq C_G(e)^c$. Therefore,

$$C_G(e) \cap N_G(T) \subseteq C_G(e)^c.$$

Now let $a \in C_G(e)$. Then T, $a^{-1}Ta$ are maximal tori of $C_G(e)^c$. So $b^{-1}a^{-1}Tab = T$ for some $b \in C_G(e)^c$. Thus $ab \in C_G(e) \cap N_G(T) \subseteq C_G(e)^c$. Since $b \in C_G(e)^c$, $a \in C_G(e)^c$.

DEFINITION. Let S be a connected monoid with group of units G and let T be a maximal torus of G.

(1) $\alpha(S) = |E(\overline{T})|$.

(2) $\beta(S) = |\mathfrak{U}(S)|$.

(3) If W is the Weyl group of G, then $\gamma(S) = |W|$.

(4) If $J \in \mathfrak{U}(S)$ and if $e \in E(J)$, then $\theta(e) = \theta(J) = |J \cap E(\overline{T})|$ and $\delta(e) = \delta(J) = \gamma(C_G(e))$.

We write θ_S, δ_S if we want to specify S.

REMARK 2.4. Since all maximal tori in G are conjugate, since all idempotents of J are conjugate and since $x^{-1}Jx = J$ for all $x \in G$, we see that the above definitions are independent of the choice of T and the particular idempotent e in J.

REMARK 2.5. If S is a connected monoid and if S_1 is a closed connected submonoid of S, then $\alpha(S_1) \leqslant \alpha(S)$. However, it can very well happen that $\beta(S_1)$ is larger than $\beta(S)$. If $e \in E(S_1)$, then $\theta_{S_1}(e) \leqslant \theta_S(e)$.

THEOREM 2.6. *Let S be a connected monoid. Then*

(1) *For all $J \in \mathfrak{U}(S)$, $\gamma(S) = \theta(J)\delta(J)$. In particular, $1 \leqslant \theta(J) \leqslant \gamma(S)$.*

(2) $\alpha(S) = \Sigma_{J \in \mathfrak{U}(S)} \theta(J)$.

(3) $\beta(S) \leqslant \alpha(S) \leqslant \beta(S)\gamma(S)$.

(4) *If $J_1, J_2 \in \mathfrak{U}_S$, then $\theta(J_1 \wedge J_2) \leqslant \theta(J_1)\theta(J_2)$.*

PROOF. Let $J \in \mathfrak{U}(S)$, T a maximal torus of G, the group of units of S. Let $A = E(\overline{T}) \cap J$. Then $\theta(J) = |A|$. Let $e \in A$. If $x \in N_G(T)$ and $f \in A$, then let $f^x = x^{-1}fx \in A$. Clearly, $(f^x)^y = f^{xy}$ for all $x, y \in N_G(T)$. So the group $N_G(T)$ acts on A. By Lemma 1.7, $N_G(T)$ acts transitively on A. The stabilizer of e is $V = C_G(e) \cap N_G(T)$. Thus $\theta(J) = |A| = |N_G(T)/V|$. Clearly $T \subseteq C_G(T) \subseteq V$. So

$$|N_G(T)/V|\,|V/C_G(T)|=|N_G(T)/C_G(T)|$$

Now V is the normalizer of T in $C_G(e)$ and $C_G(T)$ is the centralizer of T in $C_G(e)$. So $\delta(e)=|V/C_G(T)|$. Clearly $\gamma(S)=|N_G(T)/C_G(T)|$. This proves (1). Clearly $\alpha(S)=|E(\overline{T})|=\Sigma_{J\in\mathfrak{U}(S)}|J\cap E(\overline{T})|$. This yields (2). Now (3) follows from (1) and (2). Finally we prove (4). Let $h\in(J_1\wedge J_2)\cap E(\overline{T})$. Then by Lemma 1.5, there exists $e_i\in J_i\cap E(\overline{T})$ such that $e_i\geqslant h$, $i=1,2$. So $e_1e_2\geqslant h$. By Lemma 1.7, $e_1e_2=h$. So $(J_1\wedge J_2)\cap E(\overline{T})\subseteq(J_1\cap E(\overline{T}))(J_2\cap E(\overline{T}))$. Hence $\theta(J_1\wedge J_2)\leqslant\theta(J_1)\theta(J_2)$.

COROLLARY 2.7. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Then the following conditions are equivalent.*
(1) $\alpha(S)=\beta(S)$.
(2) $\theta(J)=1$ *for all* $J\in\mathfrak{U}(S)$.
(3) $\mathfrak{U}(S)\cong E(\overline{T})$.

PROOF. That $(1)\Rightarrow(2)$ follows from Theorem 2.6. Next assume (2). Define ϕ: $\mathfrak{U}(S)\to E(\overline{T})$ as $\phi(J)=e$ if $J\cap E(\overline{T})=\{e\}$. Then ϕ is clearly a bijection. That it is an isomorphism follows from Lemma 1.7. Thus $(2)\Rightarrow(3)$. That $(3)\Rightarrow(1)$ is obvious.

THEOREM 2.8. *Let S be a connected monoid with group of units G. Let e be a minimal idempotent of S. Let $S_1=\{x\mid x\in S,\ ex=xe=e\}$ and let $S_2=S_1^c$. Then*
(1) $\gamma(S)=\gamma(S_2)\gamma(eSe)$.
(2) $\mathfrak{U}(S)\cong\mathfrak{U}(S_2)$, $\alpha(S)=\alpha(S_2)$ *and* $\beta(S)=\beta(S_2)$.

PROOF. (1) By [8, Theorem 1], $eSe\subseteq\overline{C_G(e)}$. By Corollary 1.20, $\gamma(\overline{C_G(e)})=\gamma(S_2)\gamma(eSe)$. Since e lies in the kernel of S, $\theta(e)=1$. By Theorem 2.6, $\gamma(S)=\delta(e)$. This proves (1).

(2) Let G, G_2 denote the groups of units of S and S_2 respectively. Let T be a maximal torus of G with $e\in\overline{T}$. Let $T_1=\{a\mid a\in T,\ ae=e\}$, $T_2=T_1^c$. By Lemma 1.3, T_2 is a maximal torus of G_2 and $E(\overline{T})=E(\overline{T_2})$. Let $f\in E(\overline{T})$ and set $S_3=\overline{C_G(f)}$. By [8, Theorem 1], $eSe\subseteq fSf\subseteq S_3$. Then by Lemma 1.1, Theorem 2.6 and (1), $\gamma(S)=\theta_S(f)\gamma(S_3)=\theta_S(f)\gamma(S_2\cap S_3)^c\gamma(eSe)$. Also, $\gamma(S)=\gamma(S_2)\gamma(eSe)=\theta_{S_2}(f)\gamma(S_2\cap S_3)^c\gamma(eSe)$. It follows that $\theta_S(f)=\theta_{S_2}(f)$ for all $f\in E(\overline{T})$. Let $\mathfrak{C}=\{J\cap E(\overline{T})\mid J\in\mathfrak{U}(S)\}$. Since $E(\overline{T})=E(\overline{T_2})$ and since $\theta_S(f)=\theta_{S_2}(f)$ for all $f\in E(\overline{T})$, we see that $\mathfrak{C}=\{J\cap E(\overline{T})\mid J\in\mathfrak{U}(S_2)\}$. If $A_1,A_2\in\mathfrak{C}$, then define $A_1>A_2$ if there exist $f_i\in A_i$, $i=1,2$, such that $f_1>f_2$. Then by Lemma 1.7, $\mathfrak{U}(S)\cong(\mathfrak{C},\leqslant)\cong\mathfrak{U}(S_2)$. In particular $\beta(S)=\beta(S_2)$. Clearly $\alpha(S)=|E(\overline{T})|=|E(\overline{T_2})|=\alpha(S_2)$.

THEOREM 2.9. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Let S_1 be a closed connected submonoid of S such that $N_G(T)\subseteq S_1$. Then $\mathfrak{U}(S)\cong\mathfrak{U}(S_1)$, $\alpha(S)=\alpha(S_1)$ and $\beta(S)=\beta(S_1)$.*

PROOF. We see that $T\subseteq N_G(T)\subseteq S_1$. So $\overline{T}\subseteq S_1$. Let $\mathfrak{C}=\{J\cap E(\overline{T})\mid J\in\mathfrak{U}(S)\}$. If $A_1,A_2\in\mathfrak{C}$, then define $A_1<A_2$ if there exist $e_i\in A_i$, $i=1,2$, such that $e_1<e_2$. By Lemma 1.7, $\mathfrak{U}(S)\cong(\mathfrak{C},\leqslant)$. Since $N_G(T)\subseteq S_1$, we see by Lemma 1.7 that $\mathfrak{C}=\{J\cap E(\overline{T})\mid J\in\mathfrak{U}(S_1)\}$. Also, by Lemma 1.7, $\mathfrak{U}(S_1)\cong(\mathfrak{C},\leqslant)$. Hence $\mathfrak{U}(S)\cong\mathfrak{U}(S_1)$.

THEOREM 2.10. *Let S be a connected monoid and let e be a minimal idempotent of S. Then there exists a closed connected submonoid S_1 of S such that e is the zero of S_1, $e \in C(S_1)^c$, $\alpha(S) = \alpha(S_1)$, $\beta(S) = \beta(S_1)$ and $\mathfrak{U}(S) \cong \mathfrak{U}(S_1)$. In particular, $\dim C(S_1) \geqslant 1$.*

PROOF. By Theorem 2.8, we can assume that $e = 0$ is the zero of S. Let G denote the group of units of S and let T be a maximal torus of G. By [3, Chapter IX], $\Omega = N_G(T)/C_G(T)$ is a finite group. If $u \in N_G(T)$, then $a \to u^{-1}au$ is an automorphism of T. So Ω is a finite group of automorphisms of T. By Lemma 1.2, $N_G(T) = N_G(\overline{T})$ and $C_G(T) = C_G(\overline{T})$. So Ω is a finite group of automorphisms of $\overline{T}$. By Lemma 1.13, there exists a subtorus T_1 of T such that $0 \in \overline{T_1}$ and $u^{-1}au = a$ for all $a \in \overline{T_1}$ and $u \in N_G(T)$. Hence $N_G(T) \subseteq C_G(T_1)$. By [3, Theorem 22.3], $C_G(T_1)$ is a connected group. Let $S_1 = \overline{C_G(T_1)}$. Then $N_G(T) \subseteq S_1$. By Theorem 2.9, $\mathfrak{U}(S) \cong \mathfrak{U}(S_1)$. Clearly $0 \in \overline{T_1} \subseteq C(S_1)^c$.

PROBLEM 2.11. Characterize the lattice $\mathfrak{U}(S)$ where S is a connected monoid. By Theorem 2.10, the problem reduces to the case when S has zero 0 and $0 \in C(S)^c$.

See [9] for the following theorem.

THEOREM 2.12 [9]. *Let S be a connected monoid with group of units G such that the maximal subgroup of the kernel of S is solvable. Then the following conditions are equivalent:*

(1) G *is solvable.*

(2) $\alpha(S) = \beta(S)$.

(3) S *is a semilattice of archimedean semigroups.*

(4) $J^2 \subseteq J$ *for all $J \in \mathfrak{U}(S)$.*

(5) *For all $e, f \in E(S)$, any eigenvalue of ef is 0 or 1.*

THEOREM 2.13. *Let S be a connected monoid with group of units G. Let $J \in \mathfrak{U}(S)$. Then the following conditions are equivalent:*

(1) $J^2 \subseteq J$.

(2) $\theta(J) = 1$.

(3) $E(J) \subseteq \overline{B}$ *for some Borel subgroup B of G.*

(4) $E(J) \subseteq \overline{B}$ *for every Borel subgroup B of G.*

(5) *For all $e, f \in E(J)$, any eigenvalue of ef is 0 or 1.*

PROOF. (1) $\Rightarrow$ (2). Since $J^2 \subseteq J$, J is completely simple. So for any $e, f \in E(J)$, $ef = fe$ implies $e = f$. It follows that $\theta(J) = 1$.

(2) $\Rightarrow$ (3). If $e \in E(J)$, then let $X_e = \{f \mid f \in E(S), e\mathfrak{R}f\}$, $Y_e = \{f \mid f \in E(S), e\mathfrak{L}f\}$. If $f \in X_e$, then $ef = f$ and $fe = e$. If $f \in Y_e$, then $ef = e$ and $fe = f$. Now fix $e \in E(J)$. Let S_1 be a closed connected submonoid of S of smallest possible dimension such that $X_e \subseteq S_1$. Let $S_2 = \{a \mid a \in S_1, ae = e\}$. Let $f \in X_e$. Then there exists, by [7, Lemma 1.1], a closed connected submonoid S_3 of S_1 such that $f \in S_3$ and $bf = fb = f$ for all $b \in S_3$. So $S_3 \subseteq S_2$ and $S_3 \subseteq S_2^c$. So $f \in S_2^c$. Hence $X_e \subseteq S_2^c$. By the minimality of $\dim S_1$, $S_1 = S_2$. Let $e_1 \in E(S_1)$. Then by [8, Theorem 1] there exists a closed connected submonoid S' of S_1 such that $xe_1 = e_1xe_1$ for all $x \in S'$ and $e_1S_1 \subseteq S'$. Thus $X_e \subseteq S'$. By the minimality of $\dim S_1$, $S' = S_1$. So $xe_1 = e_1xe_1$

for all $e_1 \in E(S_1)$. Let $e_1, e_2 \in E(S_1)$. Then $e_1e_2e_1e_2 = e_2e_1e_2 = e_1e_2$. Thus $E(S_1)^2 \subseteq E(S_1)$. Also $eS_1e = \{e\}$. Thus the hypothesis of Theorem 2.12 is satisfied. Hence the group of units G_1 of S_1 is solvable. Now $G_1 \subseteq B_1$ for some Borel subgroup B_1 of G. Then $X_e \subseteq \bar{B}_1$. Let $f \in E(J) \cap \bar{B}_1$. Let T be a maximal torus of B_1 (and hence of G) such that $e \in \bar{T}$. By Theorem 2.1, $x^{-1}fx \in \bar{T}$ for some $x \in B_1$. Then $e, x^{-1}fx \in E(\bar{T}) \cap J$. Since $\theta(J) = 1$, $e = x^{-1}fx$. So $X_f = xX_ex^{-1} \subseteq x\bar{B}_1x^{-1} = \bar{B}_1$. Thus for all $f \in \bar{B}_1 \cap E(J)$, $X_f \subseteq \bar{B}_1$. Since $y^{-1}E(J)y = E(J)$ for all $y \in G$, and since all Borel subgroups of G are conjugate, we see that $X_f \subseteq \bar{B}$ for any Borel subgroup B of G and $f \in \bar{B} \cap E(J)$. Similarly $Y_f \subseteq \bar{B}$ for any Borel subgroup B of G and $f \in \bar{B} \cap E(J)$. Now fix a Borel subgroup B of G. Then by Theorem 2.1, there exists $e_1 \in E(J) \cap \bar{B}$. Let $e_2 \in E(J)$. Then by Lemma 1.12, there exist $e_3, e_4 \in E(J)$ such that $e_1\mathcal{R}e_3\mathcal{L}e_4\mathcal{R}e_2$. Then $e_3 \in X_{e_1} \subseteq \bar{B}$. Hence $e_4 \in Y_{e_3} \subseteq \bar{B}$. So $e_2 \in X_{e_4} \subseteq \bar{B}$. Hence $E(J) \subseteq \bar{B}$.

(3) $\Rightarrow$ (4). This follows since $X^{-1}E(J)x = E(J)$ for all $x \in G$.

(4) $\Rightarrow$ (5) follows from Theorem 2.12.

(5) $\Rightarrow$ (1) follows from Lemma 1.11.

COROLLARY 2.14. *Let S be a connected monoid with group of units G and kernel M. Then $E(M) \subseteq \bar{B}$ for every Borel subgroup B of G.*

THEOREM 2.15. *Let S be a connected monoid with group of units G. Let e be a minimal idempotent of S and let $G_0 = \{a \mid a \in G, ae = ea = e\}$. Then the following conditions are equivalent:*

(1) $\alpha(S) = \beta(S)$.

(2) S is a semilattice of archimedean semigroups.

(3) G_0^c is solvable.

(4) $E(S) \subseteq \bar{B}$ for some (every) Borel subgroup B of G.

(5) For all $e, f \in E(S)$, any eigenvalue of ef is 0 or 1.

PROOF. That (1) $\Rightarrow$ (4) follows from Corollary 2.7 and Theorem 2.13. That (4) $\Rightarrow$ (5) follows from Theorem 2.12. That (5) $\Rightarrow$ (2) $\Rightarrow$ (1) follows from Corollary 2.7, Theorem 2.13 and [9, Proposition 21]. That (1) $\Leftrightarrow$ (3) follows from Theorems 2.8 and 2.12.

The following is immediate from Theorem 2.1 and Corollary 2.7.

THEOREM 2.16. *Let S be a connected monoid such that $\alpha(S) = \beta(S)$. Then $\mathcal{U}(S)$ is a relatively complemented lattice (in fact the lattices of all faces of a polytope in some $\mathbf{R}^n$).*

CONJECTURE 2.17. By Theorems 2.10, 2.12, 2.15, 2.16, the following conjectures are equivalent:

(1) If S is a connected monoid, then $\mathcal{U}(S)$ is relatively complemented if and only if $\alpha(S) = \beta(S)$.

(2) If S is a connected monoid with zero and group of units G, then G is solvable if and only if $\mathcal{U}(S)$ is relatively complemented.

(3) If S is a connected monoid with zero 0 and group of units G, if $0 \in C(S)^c$, and if $\mathcal{U}(S)$ is relatively complemented, then G is solvable.

We now prove Conjecture 2.17 in a special case.

THEOREM 2.18. *Let S be a connected monoid such that $\mathcal{U}(S)$ is relatively complemented. If $\alpha(S) \leqslant 13$ or $\beta(S) \leqslant 7$, then S is a semilattice of archimedean semigroups.*

PROOF. By Theorem 2.15, it suffices to show that $\alpha(S) = \beta(S)$. Assume that this is not true and find a counterexample S with dim S minimal. Let G denote the group of units of S. Let $e \in E(S)$, $e \neq 1$. Then $\mathcal{U}(eSe)$ is relatively complemented by Corollary 1.9. By Lemma 1.8, $\beta(eSe) \leqslant \beta(S)$. By Lemma 1.4, $\alpha(eSe) \leqslant \alpha(S)$. By the minimality of dim S, eSe is a semilattice of archimedean semigroups. Since $\alpha(S) \neq \beta(S)$, we see by Theorem 2.6 that $\theta(J_0) > 1$ for some $J_0 \in \mathcal{U}(S)$. Choose J_0 maximal in $\mathcal{U}(S)$. Let $J' \in \mathcal{U}(S)$, $J' > J_0$. Then $\theta(J') = 1$. Let T be a maximal torus of G. Then $J' \cap E(\overline{T}) = \{\xi\}$ and there exist $e_0, e_0' \in J_0 \cap E(\overline{T})$ such that $e_0 \neq e_0'$. By Lemma 1.7, $\xi > e_0$, $\xi > e_0'$. If $\xi \neq 1$, we get a contradiction to Lemma 1.10. So $\xi = 1$ and $J' = G$. Thus G covers J_0.

Let $X = \{e \mid e \in E(\overline{T}),\ 1 \text{ covers } e\}$, $Y = \{f \mid f \in E(\overline{T}),\ e \text{ covers } f \text{ for some } e \in X\}$. Then clearly $X \cap Y = \varnothing$. By Proposition 1.14, $J_0 \cap E(\overline{T}) = J_0 \cap X$. If $e, e' \in X$, $e \neq e'$, then by Lemma 1.15, there exist distinct $e_1, \ldots, e_{k+1} \in X$, $f_1, \ldots, f_k \in Y$ such that $e = e_1$, $e' = e_{k+1}$, $e_i > f_i$, $e_{i+1} > f_i$, $i = 1, \ldots, k$. Let $\lambda(e, e')$ denote the smallest such possible k. Let $\Omega = \{(e, e') \mid e, e' \in X,\ e \neq e',\ e \,\mathcal{J}\, e'\} \neq \varnothing$. Choose $(e, e') \in \Omega$ with $\lambda(e, e') = k$ minimal. Then there exist distinct $e_1, \ldots, e_{k+1} \in X$, $f_1, \ldots, f_k \in Y$ such that $e = e_1$, $e' = e_{k+1}$, $e_i > f_i$, $e_{i+1} > f_i$, $i = 1, \ldots, k$. Let $e_i \in J_i \in \mathcal{U}(S)$, $f_i \in J_i^* \in \mathcal{U}(S)$, $i = 1, \ldots, k$. Then with $J_{k+1} = J_1$ we have $J_i > J_i^*$, $J_{i+1} > J_i$, $i = 1, \ldots, k$. Let $\mathcal{X} = \{J \mid J \in \mathcal{U}(S),\ G \text{ covers } J\}$, $\mathcal{Y} = \{J^* \mid J^* \in \mathcal{U}(S),\ \text{there exists } J \in \mathcal{X} \text{ such that } J \text{ covers } J^*\}$. Then by Proposition 1.14, $J_1, \ldots, J_k \in \mathcal{X}$ and $J_1^*, \ldots, J_k^* \in \mathcal{Y}$. We claim that $J_1, \ldots, J_k$ are distinct. For suppose that for some $\mu, \nu \in Z^+$, with $1 \leqslant \mu < \nu \leqslant k$, $J_\mu = J_\nu$. Then $e_\mu \,\mathcal{J}\, e_\nu$ and therefore $(e_\mu, e_\nu) \in \Omega$. Clearly $\lambda(e_\mu, e_\nu) < k$. This contradiction shows that $J_1, \ldots, J_k$ are distinct. Next we show that $J_1^*, \ldots, J_k^*$ are distinct. For suppose $J_\mu^* = J_\nu^*$ for some $\mu, \nu \in Z^+$ with $1 \leqslant \mu < \nu \leqslant k$. First assume $\nu > \mu + 1$. Then $J_\mu > J_\mu^*$, $J_{\mu+1} > J_\mu^*$, $J_\nu > J_\mu^*$, contradicting Proposition 1.14(5). Hence $\nu = \mu + 1$. But then $e_{\mu+1} > f_\mu$, $e_{\mu+1} > f_{\mu+1}$, $f_\mu \,\mathcal{J}\, f_{\mu+1}$, contradicting Lemma 1.10. Hence $J_1^*, \ldots, J_k^*$ are distinct. Next assume $k = 1$. Then $e_1 \,\mathcal{J}\, e_2$, $e_1 > f_1$, $e_2 > f_1$. This contradicts Proposition 1.14(7). Thus $k > 1$. Now assume $k = 2$. Then $J_\mu > J_\nu^*$ for $\mu = 1, 2$, $\nu = 1, 2$. Hence $J_1^* = J_1 \wedge J_2 = J_2^*$, a contradiction. Thus $k \geqslant 3$. Since G, $J_1, \ldots, J_k$, $J_1^*, \ldots, J_k^*$, $J_1^* \wedge J_2^*$ are distinct elements of $\mathcal{U}(S)$, $\beta(S) \geqslant 2k + 2 \geqslant 8$. Now let $e_1' = e' \in J_1$. Then $e_1' \neq e_1$. By Proposition 1.14, $J_i \cap E(\overline{T}) \subseteq X$, $J_i^* \cap E(\overline{T}) \subseteq Y$, $i = 1, \ldots, k$. By Lemma 1.5, there exists $f_1' \in J_1^* \cap E(\overline{T})$ such that $e_1' > f_1'$. If $f_1 = f_1'$, then $e_1' > f_1$, $e_1 > f_1$, $e_1 \,\mathcal{J}\, e_1'$, $e_1 \neq e_1'$. This contradicts Proposition 1.14(7). So $f_1 \neq f_1'$. Since $J_2 > J_1^*$, there exists by Lemma 1.5, $e_2' \in J_2 \cap E(\overline{T})$ such that $e_2' > f_1'$. Suppose $e_2' = e_2$. Then $e_2 > f_1$, $e_2 > f_1'$, $f_1 \,\mathcal{J}\, f_1'$, $f_1 \neq f_1'$, contradicting Lemma 1.10. Hence $e_2 \neq e_2'$. Continuing this process, we can find $e_i' \in J_i \cap E(\overline{T})$, $f_i' \in J_i^* \cap E(\overline{T})$, $i = 1, \ldots, k$, such that for any $i = 1, \ldots, k$, $e_i \neq e_i'$ and $f_i \neq f_i'$. Hence $1, e_1, \ldots, e_k, e_1', \ldots, e_k', f_1, \ldots, f_k, f_1', \ldots, f_k', f_1 f_2$ are distinct elements of $E(\overline{T})$. So $\alpha(S) \geqslant 4k + 2 \geqslant 14$. Thus $\alpha(S) \geqslant 14$ and $\beta(S) \geqslant 8$, contradicting the hypothesis of the theorem. This proves the theorem.

By Theorems 2.12, 2.16, 2.18 we have

THEOREM 2.19. *Let S be a connected monoid with zero and group of units G. Assume that either $\alpha(S) \leq 13$ or $\beta(S) \leq 7$. Then G is solvable if and only if $\mathfrak{U}(S)$ is relatively complemented.*

REMARK 2.20. The proof of Theorem 2.18 suggests that a possible counterexample to Conjecture 2.17 has the following lattice structures:

$$\mathfrak{U}(S) = \{G, J_1, J_2, J_3, J_1^*, J_2^*, J_3^*, 0\}$$

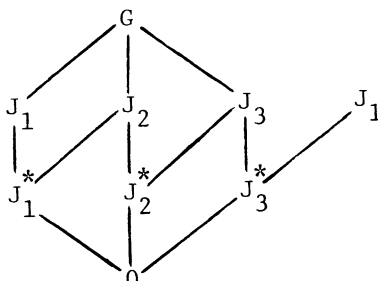with the structure illustrated in Figure 1.

FIGURE 1

$E(\overline{T}) = \{1, e_1, e_1', e_2, e_2', e_3, e_3', f_1, f_1', f_2, f_2', f_3, f_3', 0\}$, $e_i, e_i' \in J_i$, $i = 1, 2, 3$, $f_i, f_i' \in J_i^*$, $i = 1, 2, 3$ with the structure illustrated in Figure 2.
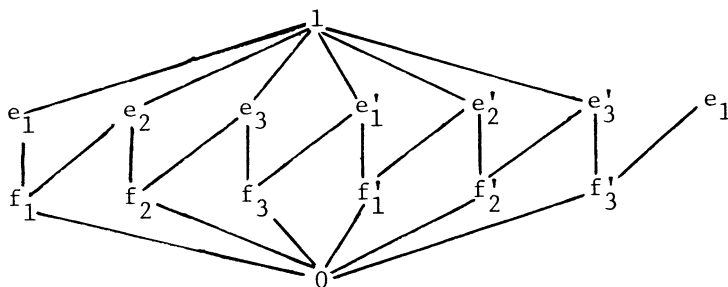

FIGURE 2

THEOREM 2.21. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Let $\mathcal{V} = \{J \mid J \in \mathfrak{U}(S), \theta(J) = 1\}$, $\Gamma = \cup \{J \cap E(\overline{T}) \mid J \in \mathcal{V}\}$. Then Γ is a subsemilattice of $(E(\overline{T}), \cdot)$, Γ is the maximal semilattice image of S, and $\Gamma \cong (\mathcal{V}, \wedge)$.*

PROOF. Let S_α ($\alpha \in \Omega$) denote the maximal semilattice decomposition of S. By Tamura [**10, 11**], each S_α is an $\mathcal{S}$-indecomposable semigroup. By [**4**], $\mathfrak{U}(S) = \cup_{\alpha \in \Omega} \mathfrak{U}(S_\alpha)$. If $\alpha \in \Omega$, then let M_α denote the kernel of S_α. By Theorem 2.13 and Lemma 1.21, $\mathcal{V} = \{M_\alpha \mid \alpha \in \Omega\}$. If $\alpha \in \Omega$, then let $e_\alpha \in E(\overline{T}) \cap M_\alpha$. Let $\alpha, \beta \in \Omega$, $S_\alpha S_\beta \subseteq S_\gamma$. Then $e_\alpha e_\beta \in S_\gamma$. Let J denote the $\mathcal{J}$-class of $e_\alpha e_\beta$. Then $J \in \mathfrak{U}(S_\gamma)$, $M_\alpha \geq J$, $M_\beta \geq J$. Let $J' \in \mathfrak{U}(S)$ such that $M_\alpha \geq J'$ and $M_\beta \geq J'$. Let $h \in J' \cap E(\overline{T})$. Then by Lemma 1.5, $e_\alpha \geq h$, $e_\beta \geq h$. So $e_\alpha e_\beta \geq h$. Thus $J \geq J'$. Hence $J = M_\alpha \wedge M_\beta$. By Theorem 2.6(4), $\theta(J) = 1$. Hence $J = M_\gamma$ and $e_\gamma = e_\alpha e_\beta$. This proves the theorem.

3. Central idempotents. In this section, we study special conditions on idempotents.

DEFINITION. Let S be a connected monoid. Then S is E-central if $E(S) \subseteq C(S)$. S is E-finite if $E(S)$ is finite. S is E-commutative if idempotents of S commute with each other.

The following theorem follows from [5, Theorem 2.8].

THEOREM 3.1 [5]. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Then the following conditions are equivalent*:

(1) *S is E-central.*

(2) *S is E-finite.*

(3) *S is E-commutative.*

(4) *$E(\overline{T}) \subseteq C(S)$.*

(5) *$E(S) = E(\overline{T})$.*

(6) *S is a semilattice of nil extensions of groups.*

If A, B are matrices, then let $A \oplus B$ denote $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. If $\mathcal{C}$, $\mathcal{B}$ are sets of matrices, then let $\mathcal{C} \oplus \mathcal{B} = \{A \oplus B \mid A \in \mathcal{C}, B \in \mathcal{B}\}$. If $a_1, \ldots, a_t \in K$, then let $\mathcal{C}(a_1, \ldots, a_t)$ denote the set of all $t \times t$ upper triangular matrices over K having $(a_1, \ldots, a_t)$ as the diagonal.

EXAMPLE 3.2. Let $r_1, \ldots, r_t \in Z^+$. Choose $\varepsilon_{ij} \in Z^+$, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r_i$. Let $w_1, \ldots, w_m$ be words in the commuting variables $X_1, \ldots, X_t$ involving each variable at least once. If $a_1, \ldots, a_t \in K$, then let $\mathcal{D}(a_1, \ldots, a_t) = \mathcal{C}(a_1^{\varepsilon_{11}}, a_1^{\varepsilon_{12}}, \ldots) \oplus \mathcal{C}(a_2^{\varepsilon_{21}}, \ldots)$ $\oplus \cdots \oplus \mathcal{C}(w_1(a_1, \ldots, a_t), \ldots, w_m(a_1, \ldots, a_t))$. Let $S = \bigcup_{a_1, \ldots, a_t \in K} \mathcal{D}(a_1, \ldots, a_t)$. Then $\overline{S}$ is a connected monoid with zero and $|E(S)| = |E(\overline{S})| = 2^t$. So $\overline{S}$ is E-central. Is $S = \overline{S}$?

CONJECTURE 3.3. Let S be an E-central connected monoid with zero. Then S is isomorphic to a closed connected submonoid of a finite direct product of semigroups of the type given in Example 3.2.

THEOREM 3.4. *Let S be a connected monoid with group of units G and let T be a maximal torus of G. Let Λ be a maximal chain in $E(\overline{T})$. If $\Lambda \subseteq C(S)$, then S is a semilattice of archimedean semigroups.*

PROOF. Let $\{1 > e_1 > \cdots > e_k\} = \Lambda$. Since $\Lambda \subseteq C(S)$, $\theta(e_i) = 1$, $i = 1, \ldots, k$. It follows by [9, Theorem 18], that $\gamma(S) = \gamma(e_k(S))$. Let $G_1 = \{a \mid a \in G, ae_k = e_k\}$. It follows from Theorem 2.8 that $\gamma(G_1^c) = 1$. Hence [3, Theorem 25.2C], G_1^c is solvable. By Theorem 2.15, S is a semilattice of archimedean semigroups.

EXAMPLE 3.5. We give an example to show that the hypothesis of Theorem 3.4 does not imply that S is E-central. Let

$$S = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & ab \end{bmatrix} \,\middle|\, a, b, c \in K \right\},$$

$$\overline{T} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{bmatrix} \,\middle|\, a, b \in K \right\}.$$

Then

$$\Lambda = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is a maximal chain in $E(\overline{T})$ and $\Lambda \subseteq C(S)$. However

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin C(S).$$

Hence S is not E-central.

THEOREM 3.6. *Let S be a connected monoid with group of units G, $\alpha(S) = n$. Then*
(1) *If S_1 is a connected E-central submonoid of S, then $|E(S_1)| \leqslant n$. If $|E(S_1)| = n$, then $S_1 \subseteq \overline{C_G(E(\overline{T}))}$ for some maximal torus of T of G.*
(2) *If T is a maximal torus of G, then $\overline{C_G(E(\overline{T}))}$ is a maximal connected E-central submonoid of S.*

PROOF. (1) Let G_1 denote the group of units of S_1, T_1 a maximal torus of G_1. Then $T_1 \subseteq T$ for some maximal torus T of G. Then $E(S_1) = E(\overline{T_1}) \subseteq C(S_1) \cap E(\overline{T})$. So $|E(S_1)| \leqslant n$. If $|E(S_1)| = n$, then $E(\overline{T_1}) = E(\overline{T})$ and $E(\overline{T}) \subseteq C(S_1)$. Then $G_1 \subseteq C_G(E(\overline{T}))$. So $S_1 \subseteq \overline{C_G(E(\overline{T}))}$.

(2) If $e \in E(\overline{T})$, then $T \subseteq C_G(e)$. Repeated application of Theorem 2.3 yields that $C_G(E(\overline{T}))$ is a connected group. Hence $\overline{C_G(E(\overline{T}))}$ is a connected E-central submonoid of S. That it is maximal follows from (1) since $\overline{T} \subseteq \overline{C_G(E(\overline{T}))}$.

EXAMPLE 3.7. Let

$$S = \left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \middle| a, b, c \in K \right\}$$

and let

$$S_0 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in K \right\},$$

$$S_j = \left\{ \begin{bmatrix} a & b \\ 0 & a^j \end{bmatrix} \middle| a, b \in K \right\} \quad \text{for } j \in Z^+.$$

Then S_i is a maximal connected E-central submonoid of S for all $i \geqslant 0$. Let G_i denote the group of units of S_i, $i = 0, 1, \ldots$. Then G_0 is abelian, G_1 is not abelian but nilpotent, G_i ($i \geqslant 2$) is not even nilpotent. Moreover S_i is not conjugate to S_j for $i \neq j$.

PROBLEM 3.8. Let S be a connected monoid. Study the maximal E-central submonoids of S.

THEOREM 3.9. *let S be a connected monoid such that for all $f \in E(S)$, the set $\{e \mid e \in E(S), e \geqslant f\}$ is finite. Then S is a semilattice of archimedean semigroups.*

PROOF. Let e be a minimal idempotent of S. Let $S_1 = \{a \mid a \in S, ae = ea = e\}$. Then S_1^c is E-finite. By Theorem 3.1, $\alpha(S_1^c) = \beta(S_1^c)$. By Theorem 2.8, $\alpha(S) = \beta(S)$. By Theorem 2.15, S is a semilattice of archimedean semigroups.

As an immediate consequence of Theorem 3.9 and [6, Theorem 3.7] we have,

THEOREM 3.10. *Let S be a connected regular monoid. Then the following conditions are equivalent*:
(1) *For all* $f \in E(S)$, *the set* $\{e \mid e \in E(S), e \geqslant f\}$ *is finite*.
(2) *S is a Clifford semigroup*.

REMARK 3.11. Theorem 3.10 is not true for connected regular semigroups (see [6, Example 4.1]).

NOTE ADDED IN THE PROOF. Theorem 2.18 is valid for any algebraic monoid with a dense group of units. Conjecture 2.17 has recently been proved by the author. See J. Algebra **73** (1981), 601–612.

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